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On the Admissibility of LPV Descriptor Systems

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Abstract.-

The contribution of this work is focused on the analysis and synthesis of stabilizing controllers based on static output feedback (SOF), for a class of descriptor linear parameter variable (LPV) systems. Descriptors systems, called also: differential-algebraic systems, singular systems, semi-state systems or generalized state-space systems; are considered to possess disturbances and parametric uncertainties of polytopic type, as are described by the following equation:

$$\mathbb{E}(\rho)\dot{z}(t) = \mathcal{F}(\rho)z(t) + \mathcal{B}(\rho)\omega(t) + Bu(t), \quad h(t) = C(\rho)z(t),$$

where ρ is a parametric variation. From a condition of existence of a linear injective application, representing the generalized inverse matrix of $\mathbb{E}$, the original descriptor system is transformed to a LPV system. Then, the condition for the static output feedback on the LPV system is analyzed. Synthesis of the SOF-based controller is obtained considering performance indices in $\mathscr{H}_2$ and $\mathscr{H}_\infty$, described as linear matrix inequalities, LMIs, as criteria in order to obtain the gain of SOF, in the presence of uncertainties and disturbances. A numerical example is presented to illustrate the results and performance of robust control.

Keywords: descriptor systems; LPV systems; static output feedback (SOF); $\mathscr{H}_2$-$\mathscr{H}_\infty$ norms; robust control.

Sobre la admisibilidad de sistemas LPV descriptores

Resumen.-

La contribución de este trabajo está centrada en el análisis y síntesis de controladores estabilizantes, basados en realimentación estática de la salida (SOF), para una clase de sistemas descriptores lineales a parámetros variantes (LPV). Los sistemas descriptores, denominados también: sistemas diferencial-algebraicos, sistemas singulares, sistemas de semi-estado o sistemas generalizados de espacio de estado; se consideran que presentan incertidumbres paramétricas de tipo politópicas y perturbaciones, tal como se describe por la ecuación siguiente:

$$\mathbb{E}(\rho)\dot{z}(t) = \mathcal{F}(\rho)z(t) + \mathcal{B}(\rho)\omega(t) + Bu(t), \quad h(t) = C(\rho)z(t),$$

donde ρ es la variación paramétrica. A partir de una condición de existencia de una aplicación inyectiva lineal, representando la inversa generalizada de la matriz $\mathbb{E}$, el sistema descriptor original es transformado a un sistema LPV. Luego, se analiza la condición para la realimentación estática de la salida sobre ese sistema LPV. La síntesis del controlador por SOF se obtiene considerando índices de desempeño en $\mathscr{H}_2$ y $\mathscr{H}_\infty$, descritos como desigualdades matriciales lineales, LMIs, como criterios para obtener la ganancia de SOF, bajo la presencia de incertidumbres y perturbaciones. Para ilustrar los resultados y el desempeño del control robusto, se presentan dos ejemplos numéricos.

Palabras clave: sistemas descriptores; sistemas LPV; realimentación estática de la salida; normas $\mathscr{H}_2$-$\mathscr{H}_\infty$; control robusto.

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1. Introduction

Since its introduction in 1977 [1], descriptor systems (DS), also called singular systems, semi-state systems, differential-algebraic systems or generalized state-space systems; have been one of the main research fields within control theory, since they are a natural and general representation of dynamic systems. Unlike their regular counterparts in state space, a DS allows a representation that incorporates algebraic constraints in their physical variables. Over the past two decades, descriptor systems have attracted much attention because of the comprehensive uses in many real world systems, such as in the economy (Leontief dynamic model), social models, electrical systems, chemical processes, and mechanical models (robotics). Considerable progress has since been made in the investigation of such systems. A problem that has been well studied is the admissibility of DS, being a research line that still remains open.

On the other hand, the context of linear parameter variable (LPV) systems refers to linear dynamical systems whose state-space representations depend on exogenous non-stationary parameters [2]. LPV systems are a generalization of LTV systems, establishing an intermediate model between linear and nonlinear dynamics, so they can be constituted in a representative model for the control of non-linear processes, allowing the use of all machinery of control of linear systems to the case of particular nonlinear processes control [3, 4]. In addition, if the nonlinear model is formulated as a parameterized linear system, where parameterization is state dependent, it allows an LPV description to represent a non-local nonlinear system, taking advantage of the consequences of a global stabilization [5, 4]. Thus, the LPV representation of a nonlinear system describes a class of systems larger than the original nonlinear system.

When there are combined the modeling of physical systems with uncertain parameters, there arise dynamic systems that represent uncertain DS. As is well known, for modeling many applications and technical processes, only approximate models are available, so that the analysis of DSs subject to uncertainties has been a very active research line. For example, numerous analysis and synthesis problems have been addressed in the literature: the analysis of robust stability (admissibility), stabilization, analysis of the robust controllability and observability, robust control under the characterization of the $\mathscr{H}_\infty$-$\mathscr{H}_2$ norms, robust filtering, analysis and positive real control, among other lines of work, [6, 7, 8, 9]. The main results in the analysis and synthesis of DS-dependent parameters are based on parametric Lyapunov functions, which allow to minimize the conservatism of classic Lyapunov functions, when searching numerical solutions through LMIs, representing a formulation that allows the resolution of complicated control problems very efficiently, and with a remarkable degree of simplicity [10, 9].

In this context, this paper addresses the analysis of robust admissibility and control for an DS class of continuous time and with polytopic type uncertainties in the dependent parameters, by using the characterizations of the $\mathscr{H}_2$-$\mathscr{H}_\infty$ norms as LMIs, which arise from parameter dependent Lyapunov functions. The DS class is the one where there is a linear injection application that allows to transform the parameter dependent DS to a regular LPV system. The existence of linear transformation ensures that the analysis of the properties results of the transformed LPV system are transferred to the finite modes of the original parameter dependent DS. Likewise, the robust control design, for the transformed LPV system, is a guarantee of satisfying the admissibility and robust performance for the original DS system. Thus, the condition for the static output feedback (SOF) on the transformed LPV system is analyzed. The SOF controller synthesis is obtained by considering performance indexes in $\mathscr{H}_2$ and $\mathscr{H}_\infty$, described as LMIs, as criteria to obtain the extended SOF gain, which considers a feedback gain for the output, and a feedback gain for its derivative.

Notation. $\mathfrak{R}$ is the set of real numbers. For a matrix A, A^T denotes its transpose. $\mathrm{tr}(A)$ defines the trace of the matrix A. In symmetric matrices partitions $\star$ denotes each of its symmetric blocks. $\mathbb{I}$ defines the identity matrix of appropriate dimension. $\mathcal{L}_2$ is the Hilbert space of vectorial signals defined on $(-\infty, \infty)$, with scalar product $< x \mid y >= \int_{-\infty}^{\infty} x(\tau)^* y(\tau) d\tau$ and such that $\| x \|_2 \triangleq < x \mid x >^{1/2} < \infty, \forall x \in \mathcal{L}_2$.

1.1. Preliminaries

Important results that must be taken into account, since they will be used in the development of the proposed technique, correspond to the extended characterizations as linear matrix inequalities (LMI)s of the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms for linear systems [11, 12].

Consider the LTI system defined by

$$\begin{aligned} \dot{x} &= Ax + B\omega \\ y &= Cx + D\omega \end{aligned} \tag{1}$$

Lemma 1 (Relaxed $\mathscr{H}_2$ performance).
Consider the system (1)*, where* $D = 0$*. For* $P = P^T > 0$*, the following statements are equivalent:*

i) *A is stable and* $\left\| C(s\mathbb{I} - A)^{-1} B \right\|_2^2 < \mu$.

ii) *There exist P and Z, such that:* $tr(Z) < 1$ *and*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -\mu\mathbb{I} \end{bmatrix} < 0, \quad \begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0 \tag{2}$$

iii) *There exist* P, Z *and G, such that:* $tr(Z) < 1$ *and*

$$\begin{bmatrix} -(G + G^T) & G^T A + P & G^T B & G^T \\ A^T G + P & -P & 0 & 0 \\ B^T G & 0 & -\mu\mathbb{I} & 0 \\ G & 0 & 0 & -P \end{bmatrix} < 0, \tag{3}$$

$$\begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0$$

iv) *There exist* P, Z *and G, such that:* $tr(Z) < 1$ *and*

$$\begin{bmatrix} -(G + G^T) & G^T A + P + G^T & G^T B \\ A^T G + P + G & -2P & 0 \\ B^T G & 0 & -\mu\mathbb{I} \end{bmatrix} < 0, \tag{4}$$

$$\begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0$$

Proof:
The equivalence between the three first statements has been shown in Theorem 3.3 of [13], which is based on projection lemma and its reciprocal version. The equivalence between *ii)* and *iv)* is shown in [11]. ■

For the stability analysis, one knows that when exists relations between the system dynamic matrix and the Lyapunov matrix, the results that are obtained are very conservative, as it is the case of systems with politopical uncertainties [14]. This situation can be resolved, in certain degree, uncoupling both matrices. In addition, the declaration *iv)* in the Lemma 1 provides an improved representation of the condition of performance in $\mathscr{H}_2$ presented in [13].

Similarly for the $\mathscr{H}_2$ case, there are some results to improve performance in $\mathscr{H}_\infty$, from improved versions of Bounded Real Lemma, as shown below.

Lemma 2 (Relaxed $\mathscr{H}_\infty$ performance).
Consider the system (1)*. For* $P = P^T > 0$ *and the matrix G, the following statements are equivalent:*

i) *A is stable and* $\left\| C(s\mathbb{I} - A)^{-1} B + D \right\|_\infty < \gamma$

ii) *There exist P, such that*

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma^2\mathbb{I} & D^T \\ C & D & -\mathbb{I} \end{bmatrix} < 0 \tag{5}$$

iii) *There exist P and G such that, for* $\tau \gg 1$

$$\begin{bmatrix} -(G + G^T) & G^T A + P + \tau G^T & 0 & G^T B \\ A^T G + P + \tau G & -2\tau P & C^T & 0 \\ 0 & C & -\mathbb{I} & D \\ B^T G & 0 & D^T & -\gamma^2\mathbb{I} \end{bmatrix} < 0 \tag{6}$$

Proof:
Conditions *i)* and *ii)* are the well known *Bounded Real Lemma*. Equivalence between *ii)* and *iii)* can be seen in [11]. ■

2. Descriptor systems and LPV systems

2.1. Descriptor systems

The DS systems, also called singular systems, semi-state systems, differential-algebraic systems or generalized state-space systems; have been one of the main fields of control theory research since its introduction in [1]. Over the last two decades, the DSs have attracted much attention due to the comprehensive uses in the economy, such as the Leontief dynamic model, in electrical systems, chemical processes, and mechanical models. Since then, considerable progress has been made in the investigation of such systems [8].

An DS is dynamically defined by

$$\begin{aligned} \mathbb{E}\dot{z}(t) &= \mathcal{F}z(t) + Bu(t), \\ h(t) &= Cz(t) \end{aligned} \quad (7)$$

where $z(t) \in \mathbb{R}^n$ is the vector of descriptor variable (instead of state vector), $\mathbb{E} \in \mathbb{R}^{m\times n}$, with $m \leq n$ and $\mathrm{rank}(\mathbb{E}) = r \leq n$ and which is called the descriptor matrix; and $\mathcal{F} \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times l}$, $C \in \mathbb{R}^{p\times n}$; the control function u belonging to $L_2(0, \tau; \mathbb{R}^l)$.

If $m = n$ and if for all $t \in [0, \tau]$, the polynomial $p(s) = \det(s\mathbb{E} - \mathcal{F})$ satisfies that $p(s) \neq 0$, it is said that the pair $(\mathbb{E}, \mathcal{F})$ is *regular*. Otherwise, it is called *singular*.

The solution and many of the properties of a free DS ($u = 0$) can be characterized in terms of the *Weierstraß* canonical form [15, 8], which allows to transform the matrix $\mathbb{E}$ into a Jordan canonical form, with a finite number of eigenvalues (finite dynamic mode), plus a nilpotent matrix, also in Jordan canonical form, representing a number of infinite eigenvalues (impulsive mode). The *nilpotency index* of the nilpotent matrix is called *system index*. If $\mathbb{E}$ is non-singular, the system is said to have zero index.

Definition 3. *Consider the system* (7)*, and be* $\kappa = deg(\det(s\mathbb{E} - \mathcal{F}))$. *If* $\kappa = r$ *is said that the DS is of* free impulse.

Thus, the DS (7) has κ finite dynamic modes, $r - \kappa$ impulsive modes, and $n - r$ non-dynamic modes.

Definition 4. *Let the DS given by* (7)*, with* $\mathbb{E}, \mathcal{F} \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{n\times l}$ *and* $C \in \mathbb{R}^{p\times n}$*. In addition, be the matrices:* $\mathbb{T}$ *and* $\mathbb{S}$*,with img* $\mathbb{T}$ = *ker* $\mathbb{E}^T$*, img* $\mathbb{S}$ = *ker* $\mathbb{E}$*.*

i) *For the triplet* $(\mathbb{E}, \mathcal{F}, B)$ *is said that the system is of stabilizable finite dynamics if rank*$[\lambda\mathbb{E} - \mathcal{F}, B] = n \; \forall\lambda \in C^+$.

ii) *For the triplet* $(\mathbb{E}, \mathcal{F}, B)$ *is said that the system is impulse controllable if rank*$[\lambda\mathbb{E}, \mathcal{F}\mathbb{S}, B] = n$.

iii) *For the triplet* $(\mathbb{E}, \mathcal{F}, B)$ *the system is said to be strongly stabilizable if* ***i)*** *and* ***ii)*** *are satisfied.*

iv) *For the triplet* $(\mathbb{E}, \mathcal{F}, C)$ *is said that the system has detectable finite dynamics if rank*$[\lambda\mathbb{E}^T - \mathcal{F}^T, C^T] = n \; \forall\lambda \in C^+$.

v) *For the triplet* $(\mathbb{E}, \mathcal{F}, C)$ *is said that the system is impulse observable if rank*$[\lambda\mathbb{E}^T, \mathcal{F}^T\mathbb{T}, C^T] = n$.

vi) *For the triplet* $(\mathbb{E}, \mathcal{F}, C)$ *the system is said to be strongly detectable if* ***iv)*** *and* ***v)*** *are satisfied.*

A controllability analysis for DS is presented in [16, 17]. In that order of ideas, in [18] a study of the controllability condition for a semilineal non-autonomous DS, by transforming the system from a linear injective application, is presented.

Theorem 5. *Let the system* (7)*, with the pair* $(\mathbb{E}, \mathcal{F})$ *regular; and let* $u = 0$.

1. *The trivial solution* $z = 0$ *of the system is stable if and only if all the finite eigenvalues of* $\lambda\mathbb{E} - \mathcal{F}$ *are in the closed left half-plane and the eigenvalues on imaginary axis are simple.*
2. *The trivial solution* $z = 0$ *of the system is asymptotically stable if and only if all the finite eigenvalues of* $\lambda\mathbb{E} - \mathcal{F}$ *are in the open left half-plane. This means that finite dynamic modes are asymptotically stable.*

Proof:
See [19, 7]. ■

The asymptotic stability of DS (7) can be characterized by a generalized projected Lyapunov equation given by:

$$\mathbb{E}^T \mathcal{X} \mathcal{F} + \mathcal{F}^T \mathcal{X} \mathbb{E} = -Q, \tag{8}$$

which has a unique solution $\mathcal{X}$ corresponding to a positive semi-definite symmetric matrix, and Q is a positive definite matrix.

Definition 6. *Consider the system* (7). *It is said that the DS is* admissible *if it is regular, free impulse and stable.*

Definition 6 allows to establish conditions for the control of DS in the sense of its stabilization [10]. Indeed:

1. For the triplet $(\mathbb{E}, \mathcal{F}, B)$ is said that the system has stabilizable finite dynamics and impulse controllable if a matrix $\mathbb{K}$ exists such that the pair $(\mathbb{E}, \mathcal{F} + B\mathbb{K})$ is admissible.
2. For the triplet $(\mathbb{E}, \mathcal{F}, C)$ is said that the system is of finite dynamics detectable and impulse observable if a matrix $\mathbb{L}$ exists such that the pair $(\mathbb{E}, \mathcal{F} + \mathbb{L}C)$ is admissible.

On the other hand, if the system (7) is regular and free impulse, through the algebraic-differential manipulation of the non-dynamic modes, it is possible to obtain a system descriptor of the form

$$\begin{aligned} \mathbf{E}\dot{z}(t) &= \mathbb{F}z(t) + \mathbb{B}u(t), \\ h(t) &= Cz(t) \end{aligned} \tag{9}$$

where $\mathbf{E} \in \Re^{r\times n}$, $\mathbb{F} \in \Re^{r\times n}$ and $\mathbb{B} \in \Re^{r\times m}$, representing the dynamic modes of the original system. Thus, the admissibility problem of the system (7) is equivalent to the admissibility of the system (9).

For example, be the descriptor system

$$\begin{bmatrix} e_1 & e_2 \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \omega \tag{10}$$

with $e_1, e_2 \neq 0$. This system is regular and free impulse, with $r = 1$. Thus, from the non-dynamic modes we obtain the DS given by

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} \dot{z} = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 \end{bmatrix} z + u \tag{11}$$

for which the admissibility problem corresponds to the admissibility of the original DS. The procedure is extended to higher order DS.

2.2. *LPV systems*

The context of LPV systems refers to linear dynamical systems whose state space representations depend on non-stationary exogenous parameters [2]. LPV systems are a generalization of LTV systems, establishing an intermediate model between linear and nonlinear dynamics, so that they can be constituted in a representative model for the control of non-linear processes, allowing the use of all machinery to control linear systems to the case of to design controllers for particular nonlinear processes [3, 4].

Definition 7. *An LPV is a dynamic system in which matrices contain functions that depend on a vector of known variant parameters.*

According to Definition 7, a representative LPV model is of the form:

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t); \quad x(t_0) = x_0 \\ y(t) &= C(\rho(t))x(t) \end{aligned} \tag{12}$$

where $x(t) \in \Re^n$ are the states, $u(t) \in \Re^p$ are the controls and $y(t) \in \Re^q$ are the measured output. $\rho(t) : \mathbb{R}_+ \to \mathbb{R}^l$. If $\rho(t) = t$, $l = 1$, the LPV model describes an LTV system.

From a practical point of view, an LPV system has at least two interesting interpretations [20]:

1. It can be seen as a LTI system with parametric uncertainty, where the uncertainty is given by the time variant parameter $\rho(t)$.
2. It can be seen as a LTV model, or a model resulting from the linearization of a nonlinear system (NLS) along the trajectories of the parameter ρ, applying extended linearization or based-on velocity linearization, which allows to apply the very well known methods and techniques of analysis and synthesis of linear systems.

The typical constraints on exogenous parameters are limits on magnitudes and their indexes of variations, that is, $\forall t \geq 0$

$$\underline{\rho} \leq \rho(t) \leq \bar{\rho}, \qquad \underline{\mu} \leq \dot{\rho}(t) \leq \bar{\mu} \tag{13}$$

2.2.1. Polytopical LPV systems

Consider the system (12). That system can be characterized as a polytope if it is defined

$$\mathcal{P} := \begin{pmatrix} A(\rho) & B(\rho) \\ C(\rho) & 0 \end{pmatrix} \in \Omega. \tag{14}$$

where Ω is a polytopic set, which is defined as:

$$\Omega := \left\{ \mathcal{P} : \mathcal{P} = \sum_{i=1}^{l} \rho_i \mathcal{P}_i;\ \rho_i \geq 0;\ \sum_{i=1}^{l} \rho_i = 1 \right\} \tag{15}$$

so that any admissible matrix $\mathcal{P}$ of the system can be written as an unknown convex combination of l vertex matrices given, such that

$$\mathcal{P}_i = \begin{pmatrix} A_i & B_i \\ C_i & 0 \end{pmatrix} \tag{16}$$

where A_i, B_i, C_i, $i = 1, \ldots, l$, are given matrices, representing the polytope vertices. Thus, this system can be characterized by the convex hull of Ω considering the vertices of the polytope, i.e.

$$C_o\, \Omega = \left\{ \begin{pmatrix} A_1 & B_1 \\ C_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_l & B_l \\ C_l & 0 \end{pmatrix} \right\}. \tag{17}$$

where these matrix vertices are known, provided that $\rho_i \in \Re,\ \ \rho_i \geq 0,\ \ i = 1, \ldots, l,\ \ \sum_{i=1}^{l} \rho_i = 1$.

Consequently, from the dependence of the system matrices with respect to the ρ parameter, and from the membership of those matrices to the polytope Ω, then, with $x(t_0) = x_0$:

$$\begin{aligned} \dot{x}(t) &= \left(\sum_{i=1}^{l} A_i \rho_i \right) x(t) + \left(\sum_{i=1}^{l} B_i \rho_i \right) u(t); \\ y(t) &= \left(\sum_{i=1}^{l} C_i \rho_i \right) x(t) \end{aligned} \tag{18}$$

where $\rho_i \in \Re,\ \ \rho_i \geq 0,\ \ i = 1, \ldots, l,\ \ \sum_{i=1}^{l} \rho_i = 1$.

The controllability and observability conditions of these systems can be analyzed in [21, 5] and [22]. The stability and robust stabilization of polytopic LPV systems can be studied in [23, 2], as well in [24, 22].

3. Problem formulation

Consider an DS as (7), but with parametric uncertainty and perturbations, that is:

$$\begin{aligned} \mathbb{E}(\rho)\dot{z}(t) &= \mathcal{F}(\rho)z(t) + \mathcal{B}_1(\rho)\omega(t) + Bu(t) \\ h(t) &= C(\rho)z(t) + \mathcal{D}_1(\rho)\omega(t) \\ y(t) &= C_2 z(t) \end{aligned} \tag{19}$$

which constitutes an LPV descriptor system. There, $u(t) \in \Re^q$ are the controls; $\omega(t) \in \Re^d$ are disturbances; $h(t) \in \Re^p$ are the controlled outputs and $y(t) \in \Re^p$ are the measured outputs. The parametric variation ρ is assumed to meet the constraints defined in (13). $\mathbb{E} \in \mathbb{R}^{m\times n}$, and for all ρ, $\text{rank}(\mathbb{E}) = r < n$. It can be assumed that $r = m$, so that the system is of the form given by (9), with parametric uncertainties. The matrices $\mathcal{F}$, $\mathcal{B}_1$, B, C, $\mathcal{D}$ and C_2 are of appropriate dimensions. In addition, for all ρ, it is assumed that:

1. For the triplet $(\mathbb{E}(\rho), \mathcal{F}(\rho), B)$, the finite dynamics of the system is stabilizable and impulse controllable.
2. For the triplet $(\mathbb{E}(\rho), \mathcal{F}(\rho), C_2)$, the finite dynamics of the system is detectable and impulse observable.

The above conditions lead to solutions to the stability and robust performance problem for the system (19). B and C_2 are matrices known for the fact that they characterize, from a practical point of view, the actuators and sensors, respectively, which are the suitably selected devices in the control systems.

The study of the robust stabilization of DS type LPV has been reported in [6, 15, 25, 26, 27]. In these contributions the controllers are state feedback type and the uncertainty is usually assumed only in the dynamic matrix. In [28, 29] and in [30] output feedback is applied, again in models with very particular uncertainties. Finally, in [10, 9] SOF is used for the robust stabilization of a polytopic type DS with the matrix $\mathbb{E}$ certain and undisturbed.

3.1. Robust stabilization and performance problem

Consider the system (19) with, for all ρ, $(\mathbb{E}(\rho), \mathcal{F}(\rho), B)$ defining an stabilizable and impulse controllable finite dynamics; and $(\mathbb{E}(\rho), \mathcal{F}(\rho), C_2)$ is such that the finite dynamics is detectable and impulse observable.

Problem 8. *Design a control $u(t)$ for the system* (19) *such that the corresponding closed loop system will be **admissible**.*

Problem 9. *Design a control $u(t)$ for the system* (19) *such that the corresponding closed loop system will be **admissible** and the effect of the perturbation $\omega(t)$ on the controlled output $h(t)$ will be minimum in the sense of the $\mathscr{H}_2$-$\mathscr{H}_\infty$ norms.*

For Problem 8, relative to robust stabilization, it is assumed that $\omega(t) = 0$. Next, the problem 9 demands, in addition to robust stabilization, to satisfy a robust performance index characterized under the $\mathscr{H}_2$-$\mathscr{H}_\infty$ norms.

4. Main rasults: LPV descriptor systems control

In this section we present the main results of the work, which consists in proposing an extended SOF control that depends on the output and its derivative. So, be the control of the form:

$$u(t) = \mathcal{K}_0 y(t) + \mathcal{K}_1 \dot{y}(t), \tag{20}$$

where $\mathcal{K}_0$ y $\mathcal{K}_1$ are the feedback gains, to be determined, for the output and its derivative. In this case, the derivative of the output is used in the context of the derivative action on PID controllers. Thus, the control will be given by

$$u(t) = \mathcal{K}_0 y(t) + \mathcal{K}_1 C_2 \dot{z}(t) \tag{21}$$

There are some aspects that determine the advantages of this type of controller [24]:

1. If $C_2 = \mathbb{I}$, the design is reduced to a typical state feedback control.
2. If $\mathcal{K}_1 = 0$, corresponds to a classic SOF control.
3. By using the $\mathcal{K}_0$ and $\mathcal{K}_1$ gains, many systems that can not be controlled by a classic SOF, can be stabilized by this way. In addition, it is easier to implement than a dynamic output feedback control.

As can be seen in the equation (21), the control $u(t)$ depends on the dynamics of $z(t)$. In order to construct the control, a particular class of linear DSs to variant parameters is assumed, those in which the following condition is satisfied:

$$\forall \rho, \quad \det\left(\mathbb{E}(\rho)\mathbb{E}^T(\rho)\right) \neq 0 \tag{22}$$

This means that there is a linear injective application $\Gamma(\rho)$, which represents the generalized inverse of $\mathbb{E}(\rho)$, that is, $\mathbb{E}(\rho)\Gamma(\rho)\mathbb{E}(\rho) = \mathbb{E}(\rho)$.

Therefore, let the change of variable $z(t) = \Gamma(\rho)x(t)$. Consequently, $\mathbb{E}(\rho)\Gamma(\rho) = \mathbb{I}$, then the system (19) is transformed into an LPV system:

$$\begin{aligned} \dot{x}(t) &= A(\rho)x(t) + \mathcal{B}_1(\rho)\omega(t) + Bu(t) \\ h(t) &= C_1(\rho)x(t) + \mathcal{D}_1(\rho)\omega(t) \\ y(t) &= C_\Gamma(\rho)x(t) \end{aligned} \tag{23}$$

where

$$A(\rho) = \mathcal{F}(\rho)\Gamma(\rho) - \mathbb{E}(\rho)\frac{\partial\Gamma(\rho)}{\partial\rho}\dot{\rho},$$

which is a parametric matrix that depends on ρ, $\dot{\rho}$ that, according to LPV systems, are bounded parameters; $C_1(\rho) = C(\rho)\Gamma(\rho)$ and $C_\Gamma(\rho) = C_2\Gamma(\rho)$. The control design for the original system (19) can be constructed from the transformed system (23).

Proposition 10. *Consider the system* (23). *If for all ρ the pair $(A(\rho), B)$ is controlable and the pair $(C_\Gamma(\rho), A(\rho))$ is observable, then the system* (19) *is of finite dynamic stabilizable and impulse controlable, and of finite dynamic detectable and impulse observable.*

In fact, the existence of the linear transformation $\Gamma(\rho)$ implies that the system (19) is regular and impulse free: if there exists $\Gamma(\rho)$, the regular system (23) is obtained, since the regularity of (19) depends on the pair $(s\mathbb{E}(\rho), \mathcal{F}(\rho))$ be regular, whose condition becomes the regularity of the pair $(s\mathbb{I}, A(\rho))$, which is always satisfied. In addition, (23) is characterized by the dynamic matrix $A(\rho)$ whose spectrum defines the finite modes of the system (19). Then, according to the results shown in [18], the controllability (observability) properties of the original system are transferred in the transformed system, that is, if for all ρ:

1. the triplet $(\mathbb{E}(\rho), \mathcal{F}(\rho), B)$ defines a system with finite dynamics stabilizable and impulse controllable, then the pair $(A(\rho), B)$ is controllable;
2. the triplet $(\mathbb{E}(\rho), \mathcal{F}(\rho), C_2)$ defines a system with detectable and impulse observable finite dynamics, then pair $(C_\Gamma(\rho), A(\rho))$ is observable.

Following the DS example given by (11), in this case

$$\Gamma = \begin{bmatrix} \frac{e_1}{e_1^2+e_2^2} \\ \frac{e_2}{e_1^2+e_2^2} \end{bmatrix}$$

and the transformed system will be

$$\dot{x} = \frac{e_1(a_1 - b_1) + e_2(a_2 - b_2)}{e_1^2 + e_2^2} x + u$$

Therefore, the problem of admissibility for LPV type DS becomes a problem of control of LPV systems. Consequently, let the system (23) and consider a control law given by the equation (20), then

$$u(t) = \mathbb{M}^{-1} \left(\mathcal{K}_0 C_\Gamma(\rho) + \mathcal{K}_1 C_\Gamma(\rho) A(\rho) \right) x(t), \quad (24)$$

where $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 C_\Gamma(\rho) B$.

As can be observed, the existence of the control depends on the invertibility of the matrix $\mathbb{M}$, which is a more weak condition with respect to the conditions for the typical SOF control [24, 22]. In short, the admissibility problem of the system (19) corresponds to the synthesis of a control for the system (23).

4.1. Robust Stabilization

Let the system (23) with the pair $(A(\rho), B)$ controllable and $\omega(t) = 0$. It is also assumed that the system supports a polytopic representation according to (16). Be a control of the form (24), then the closed loop dynamic matrix is:

$$A_c = A(\rho) + B\mathbb{M}^{-1} \left(\mathcal{K}_0 C_\Gamma + \mathcal{K}_1 C_\Gamma A(\rho) \right),$$

where $\mathbb{M}$ is a nonsingular matrix, so that $\mathbb{M}^{-1}$ exists, which allows to calculate $u(t)$.

Theorem 11. *Let the system* (23) *with the pair* $(A(\rho), B)$ *controllable. There is an extended SOF control of the form* (24) *that stabilizes in closed-loop system, if there exists* $\mathbb{M}$ *non-singular and the matrix* $P = P^T > 0$*, and matrices* X*,* Y*,* Z *such that the following LMI is satisfied*

$$\begin{aligned} &PA_i + A_i^T P + BXC_{\Gamma_i} + C_{\Gamma_i}^T X^T B^T + \\ &\quad BYC_{\Gamma_i} A_i + A_i^T C_{\Gamma_i}^T Y^T B^T < 0, \end{aligned} \quad (25)$$

where A_i*,* C_{Γ_i}*,* $i = 1, \ldots, l$*, representing the polytope vertices, then the feedback gains are obtained from*

$$\mathcal{K}_0 = \mathbb{M} Z^{-1} X \quad (26)$$

$$\mathcal{K}_1 = \mathbb{M} Z^{-1} Y \quad (27)$$

with $PB = BZ$ *and* $\mathbb{M}^{-1} = \mathbb{I} + Z^{-1} Y C_\Gamma(\rho) B$*.*

<u>**Proof:**</u>
It is known that for closed-loop stability, there exists $P = P^T > 0$ such that $PA_c + A_c^T P < 0$, then substituting

$$\begin{aligned} &PA_i + PB\mathbb{M}^{-1}\mathcal{K}_0 C_{\Gamma_i} + PB\mathbb{M}^{-1}\mathcal{K}_1 C_{\Gamma_i} A_i + A_i^T P + \\ &C_{\Gamma_i}^T \mathcal{K}_0^T (\mathbb{M}^{-1})^T B^T P + A_i^T C_{\Gamma_i}^T \mathcal{K}_1^T (\mathbb{M}^{-1})^T B^T P < 0 \end{aligned}$$

For the linearization of the matrix inequality, if $PB = BZ$ and variable changes $X = Z\mathbb{M}^{-1}\mathcal{K}_0$, $Y = Z\mathbb{M}^{-1}\mathcal{K}_1$, the LMI given by (25) is obtained. Moreover, as $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 C_\Gamma(\rho) B$ and $Z^{-1} Y = \mathbb{M}^{-1}\mathcal{K}_1$, then the expression for $\mathbb{M}^{-1}$ is obtained, which depends on the numerical solution of the LMI (25), on known matrices of the system (19), and on $C_\Gamma(\rho)$, which can be selected from the central value of ρ. ■

It may be noted that $Z^{-1} = (B^T B)^{-1} B^T P^{-1} B$. Thus, the admissibility problem of the system (19) is solved by robust stabilization of the system (23).

4.2. Robust stabilization and performance

Let the system (23) with the pair $(A(\rho), B)$ controllable. This system supports a polytopic representation according to (16). Be a control given by (24), then the closed loop system is:

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + \mathcal{B}_c \omega(t) \\ h(t) &= C_1(\rho) x(t) + \mathcal{D}_1(\rho) \omega(t) \end{aligned} \quad (28)$$

where

$$\begin{aligned} A_c &= A(\rho) + B\mathbb{M}^{-1}\mathcal{K}_0 C_\Gamma(\rho) + \\ &\quad B\mathbb{M}^{-1}\mathcal{K}_1 C_\Gamma(\rho)A(\rho) \\ \mathcal{B}_c &= \mathcal{B}_1(\rho) + B\mathbb{M}^{-1}\mathcal{K}_1 C_\Gamma(\rho)\mathcal{B}_1(\rho) \end{aligned}$$

As it has been proposed, it is required to design $\mathcal{K}_0$ and $\mathcal{K}_1$ such that A_c be robustly stable, and that the effect of the perturbation $\omega(t)$ on controlled output $h(t)$ be minimum according to performance indices in $\mathscr{H}_2 - \mathscr{H}_\infty$, which are characterized by LMIs according to the Lemma 1 and the Lemma 2.

4.2.1. Design in $\mathscr{H}_2$

Theorem 12. *Let the system* (23) *with the pair* $(A(\rho), B)$ *controllable and* $\mathcal{D}_1(\rho) = 0$*, which supports a polytopic representation whose vertices are defined by* A_i, $\mathcal{B}_{1_i}$, C_{Γ_i} *and* C_{1_i}*. There is a control law of the form* (24)*, which guarantees a suboptimal performance in* $\mathscr{H}_2$ *for the closed loop system* (28)*, if there exist* $G \in \Re^{n\times n}$, $X \in \Re^{n\times p}$, $Y \in \Re^{n\times p}$, $P_i = P_i^T > 0 \in \Re^{n\times n}$, $W \in \Re^{p\times n}$ *such that* $tr(W) < 1$ *and the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & \Phi & \Upsilon \\ \star & -2P_i & 0 \\ \star & \star & -\mu\mathbb{I} \end{bmatrix} < 0, \begin{bmatrix} P_i & (C_{1_i})^T \\ C_{1_i} & W \end{bmatrix} > 0, \tag{29}$$

for $i = 1, \ldots, l$*, where* $\Phi = G^T A_i + BXC_{\Gamma_i} + BYC_{\Gamma_i}A_i + P_i + G^T$ *and* $\Upsilon = G^T\mathcal{B}_{1_i} + BYC_{\Gamma_i}\mathcal{B}_{1_i}$*. The feedback gains are:*

$$\mathcal{K}_0 = \mathbb{M}Z^{-1}X \tag{30}$$

$$\mathcal{K}_1 = \mathbb{M}Z^{-1}Y \tag{31}$$

with $G^T B = BZ$ *and* $\mathbb{M}^{-1} = \mathbb{I} + Z^{-1}YC_\Gamma(\rho)B$*.*

Proof:

Applying clause iv) of the Lemma 1 to the closed loop system (28), matrix inequalities are obtained. After, for the matrix inequalities linearization are used the changes of variables $G^T B = BZ$ and $X = Z\mathbb{M}^{-1}\mathcal{K}_0$, $Y = Z\mathbb{M}^{-1}\mathcal{K}_1$, which generate, by substitution, the LMI (29). ■

In this case, $Z^{-1} = (B^T B)^{-1}B^T(G^T)^{-1}B$, so that the gains are obtained from the numerical solution of the LMI and known matrices of the original system (19). Consequently, the admissibility with robust performance of the system (19), has been solved in the transformed system as a robust control problem in $\mathscr{H}_2$, using extended SOF.

4.3. Design in $\mathscr{H}_\infty$

Theorem 13. *Let the system* (23) *with the pair* $(A(\rho), B)$ *controllable, which supports a polytopic representation whose vertices are defined by* A_i, $\mathcal{B}_{1_i}$, C_{Γ_i} *and* C_{1_i}*. There is a control law of the form* (24)*, which guarantees a suboptimal performance in* $\mathscr{H}_\infty$ *for the closed loop system* (28)*, if the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & \Phi & 0 & \Upsilon \\ \star & -2\tau P_i & (C_{1_i})^T & 0 \\ \star & \star & -\mathbb{I} & \mathcal{D}_{1_i} \\ \star & \star & \star & -\gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{32}$$

for $i = 1, \ldots, l$*, where* $\Phi = G^T A_i + BXC_{\Gamma_i} + BYC_{\Gamma_i}A_i + P_i + \tau G^T$ *and* $\Upsilon = G^T\mathcal{B}_{1_i} + BYC_{\Gamma_i}\mathcal{B}_{1_i}$*;* $G \in \Re^{n\times n}$, $X \in \Re^{n\times p}$, $Y \in \Re^{n\times p}$, $P_i = P_i^T > 0 \in \Re^{n\times n}$ *and* $\tau >> 1$*. The feedback gains are:*

$$\mathcal{K}_0 = \mathbb{M}Z^{-1}X \tag{33}$$

$$\mathcal{K}_1 = \mathbb{M}Z^{-1}Y \tag{34}$$

with $G^T B = BZ$ *and* $\mathbb{M}^{-1} = \mathbb{I} + Z^{-1}YC_\Gamma(\rho)B$*.*

Proof:

Considering the closed-loop system (28), the Lemma 2 is applied. Then, the procedure of linearization of matrix inequalities is followed by changes of variables, as has been done for the proof of the Theorem 12. ■

In order to reduce conservatism, in the characterization of the relaxed norms in $\mathscr{H}_2$-$\mathscr{H}_\infty$ as LMIs, the P matrix does not necessarily have to be unique, so that the matrices $P_i = P_i^T > 0$ can be used. On the other hand, mixed performance indices in $\mathscr{H}_2$-$\mathscr{H}_\infty$ can be imposed, so control synthesis, for robust admissibility and performance in closed loop, meet multiple objectives.

4.4. Design using DOF

In order to avoid the selection of $C_\Gamma(\rho)$, from the central value of ρ, the controller design can be followed by dynamic feedback output (DOF). In this case, a dynamic controller is proposed, such as:

$$\begin{aligned} \dot{\zeta}(t) &= \mathcal{A}_k\zeta(t) + \mathcal{B}_k y(t) \\ u(t) &= C_k\zeta(t) + \mathcal{D}_k y(t) \end{aligned} \tag{35}$$

Therefore, for system (23) and the control (35), the closed loop system is given by

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{\zeta}(t) \end{pmatrix} &= \begin{pmatrix} A(\rho) + B\mathcal{D}_k C_\Gamma(\rho) & BC_k \\ \mathcal{B}_k C_\Gamma(\rho) & \mathcal{A}_k \end{pmatrix} \begin{pmatrix} x(t) \\ \zeta(t) \end{pmatrix} + \\ & \quad \begin{pmatrix} \mathcal{B}_1(\rho) \\ 0 \end{pmatrix} \omega(t) \\ h(t) &= C_1(\rho)x(t) + \mathcal{D}_1(\rho)\omega(t) \\ y(t) &= C_\Gamma(\rho)x(t) \end{aligned} \tag{36}$$

The dynamic matrix of the closed loop system is described by

$$\mathcal{A}_c = \mathfrak{A}(\rho) + \mathfrak{B}\mathfrak{K}_0\mathfrak{C}(\rho) \tag{37}$$

where

$$\mathfrak{A}(\rho) = \begin{pmatrix} A(\rho) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} B & 0 \\ 0 & \mathbb{I} \end{pmatrix},$$

$$\mathfrak{C}(\rho) = \begin{pmatrix} C_\Gamma(\rho) & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad \mathfrak{K}_0 = \begin{pmatrix} \mathcal{D}_k & C_k \\ \mathcal{B}_k & \mathcal{A}_k \end{pmatrix}.$$

Thus, $\mathfrak{K}_0$ can be seen as design gain for a SOF control problem for the system defined by the $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ matrices. Consequently, Theorem 11 can be applied for robust stabilization. Theorem 12 or Theorem 13 can be used, in order to obtain the gain $\mathfrak{K}_0$ for robust stabilization and performance.

Corollary 14 (Robust admisibility). *Let the system* (23) *with the pair* $(A(\rho), B)$ *controllable. There is a DOF control of the form* (35) *that stabilizes the closed-loop system, if there exists the matrix* $P = P^T \succ 0$, *and matrices* X, Z *such that the following LMI is satisfied*

$$P\mathfrak{A}_i + \mathfrak{A}_i^T P + \mathfrak{B}X\mathfrak{C}_i + \mathfrak{C}_i^T X^T \mathfrak{B}^T \prec 0, \tag{38}$$

where $\mathfrak{A}_i$, $\mathfrak{C}_i$, $i = 1, \ldots, l$, *representing the polytope vertices, then the feedback gain are obtained from*

$$\mathfrak{K}_0 = Z^{-1}X \tag{39}$$

with $P\mathfrak{B} = \mathfrak{B}Z$.

Corollary 15 (Robust admisibility in $\mathscr{H}_2$). *Let the system* (23) *with the pair* $(A(\rho), B)$ *controllable and* $\mathcal{D}_1(\rho) = 0$, *which supports a polytopic representation whose vertices are defined by* A_i, $\mathcal{B}_{1_i}$, C_{Γ_i} *and* C_{1_i}. *There is a DOF control of the form* (24), *which guarantees a suboptimal performance in* $\mathscr{H}_2$ *for the closed loop system* (36), *if there exists* G, X, $P_i = P_i^T \succ 0$, W *such that* $tr(W) < 1$ *and the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & \Phi & \Upsilon \\ \star & -2P_i & 0 \\ \star & \star & -\mu\mathbb{I} \end{bmatrix} < 0,$$
$$\begin{bmatrix} P_i & (\mathfrak{C}_{1_i})^T \\ \mathfrak{C}_{1_i} & W \end{bmatrix} > 0, \tag{40}$$

for $i = 1, \ldots, l$, *where* $\mathfrak{B}_{1_i} = \begin{pmatrix} \mathcal{B}_{1_i} \\ 0 \end{pmatrix}$, $\mathfrak{C}_{1_i} = \begin{pmatrix} C_{1_i} & 0 \end{pmatrix}$, $\Phi = G^T\mathfrak{A}_i + \mathfrak{B}X\mathfrak{C}_i + P_i + G^T$ *and* $\Upsilon = G^T\mathfrak{B}_{1_i}$. *The feedback gain is:*

$$\mathfrak{K}_0 = Z^{-1}X \tag{41}$$

with $G^T\mathfrak{B} = \mathfrak{B}Z$.

Corollary 16 (Robust admisibility in $\mathscr{H}_\infty$). *Let the system* (23) *with the pair* $(A(\rho), B)$ *controllable, which supports a polytopic representation whose vertices are defined by* A_i, $\mathcal{B}_{1_i}$, C_{Γ_i} *and* C_{1_i}. *There is a DOF control of the form* (24), *which guarantees a suboptimal performance in* $\mathscr{H}_\infty$ *for the closed loop system* (36), *if there exists* G, X, $P_i = P_i^T \succ 0$ *such that the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & \Phi & 0 & \Upsilon \\ \star & -2\tau P_i & (\mathfrak{C}_{1_i})^T & 0 \\ \star & \star & -\mathbb{I} & \mathcal{D}_{1_i} \\ \star & \star & \star & -\gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{42}$$

for $i = 1, \ldots, l$, *where* $\mathfrak{B}_{1_i} = \begin{pmatrix} \mathcal{B}_{1_i} \\ 0 \end{pmatrix}$, $\mathfrak{C}_{1_i} = \begin{pmatrix} C_{1_i} & 0 \end{pmatrix}$, $\Phi = G^T\mathfrak{A}_i + \mathfrak{B}X\mathfrak{C}_i + P_i + G^T$, $\Upsilon = G^T\mathfrak{B}_{1_i}$, *and* $\tau >> 1$. *The feedback gain is:*

$$\mathfrak{K}_0 = Z^{-1}X \tag{43}$$

with $G^T\mathfrak{B} = \mathfrak{B}Z$.

For all cases, the matrices of the dynamic controller $\mathcal{A}_k, \mathcal{B}_k, C_k, \mathcal{D}_k$, are determined from $\mathfrak{K}_0$.

5. Numerical example

5.1. Parametric DS

Consider the dynamic modes of one DS described by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \end{bmatrix}\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -(1+\theta) & 1 & 0 \\ 1 & 0 & -(1+\theta) \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\omega$$

$$y = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{44}$$

where $\theta \in [1 \ \ 2]$ and $\dot\theta \in [-1 \ \ 1]$. For this case,

$$\mathbb{E}(\theta)\mathbb{E}^T(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & \theta^2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\theta} \\ 0 & 0 \end{bmatrix},$$

resulting the following LPV system:

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} -(1+\theta) & \frac{1}{\theta} \\ 1 & \frac{\dot\theta}{\theta} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\omega$$

$$y = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{\theta} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{45}$$

It can easily be verified that, for all $\theta, \dot\theta$, this system is controllable, which means that the DS can be admissible. In addition, for the stabilization, the transformed system admits an SOF control, so that Theorem 13 can be applied, only to determine the gain $\mathcal{K}_0$, which is sufficient for the robust stabilization of the closed-loop LPV system. From the numerical solution of the LMI, with γ = 0,0227, is obtained:

$$G^T = \begin{bmatrix} 0{,}2038 & -0{,}2025 \\ -0{,}1971 & 0{,}2221 \end{bmatrix},$$

$$X = [-10{,}2933 \quad 11{,}1305];$$

thus $Z = 0{,}2221$, and

$$\mathcal{K}_0 = [-46{,}3432 \quad 50{,}1127].$$

The Figure 1 shows the closed-loop poles as a function of the parameter θ, $\dot\theta$, which, as can be seen, are stable (see Figure 2). Consequently, the robust admissibility of the DS is guaranteed.

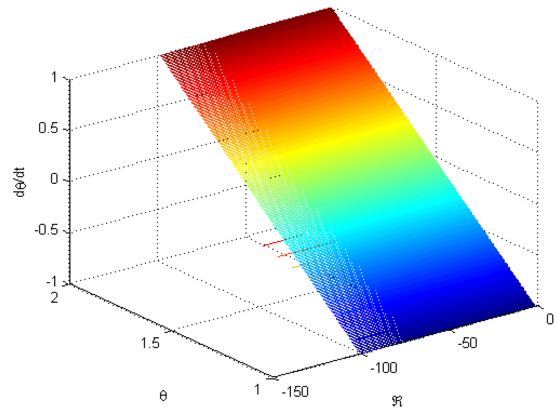

Figure 1: Distribution of closed loop poles.

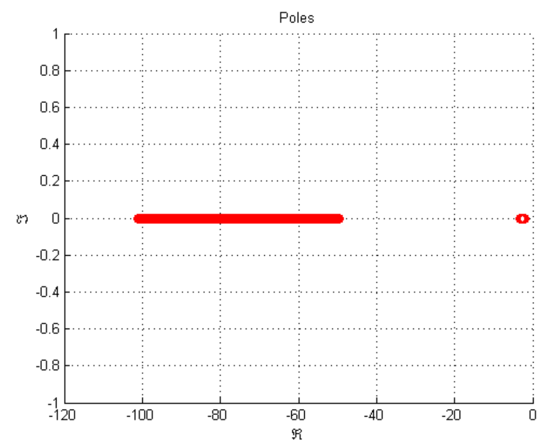

Figure 2: Distribution of projected poles of the closed loop.

To evaluate the outputs of the controlled system, the system in closed loop has been simulated with parametric variations using Matlab®-Simulink®, as is shown in the Figure 3.

Applying the robust control by SOF, which is shown in the Figure 4 and due to robust admissibility, it can be seen that the outputs converge to their steady state, such as is shown in the Figure 5, which is only affected by the perturbation $\omega(t)$, the signal that is shown in the Figure 6.

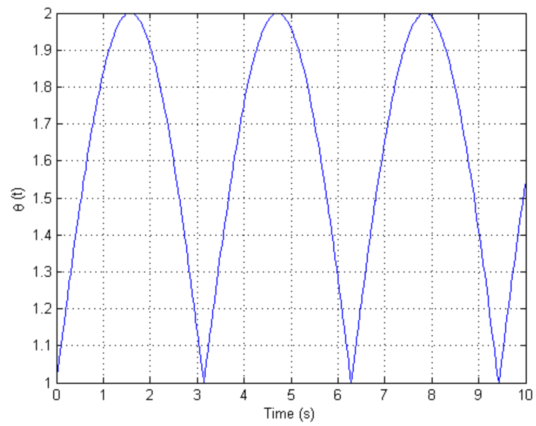

Figure 3: Parameter behavior θ.

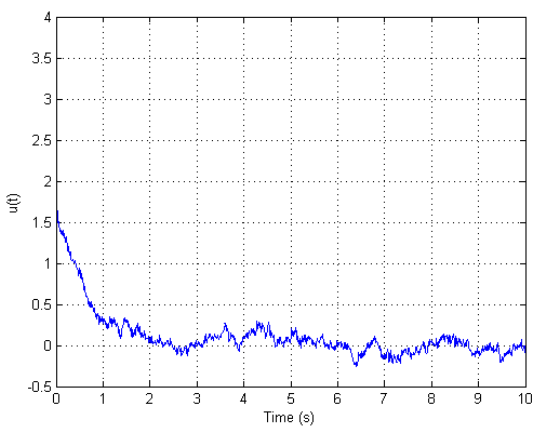

Figure 4: Control temporal behavior.

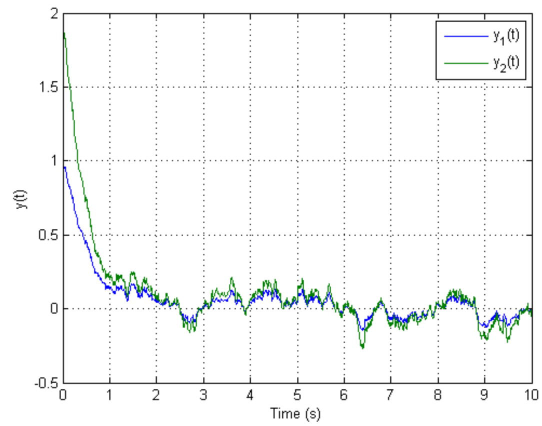

Figure 5: Temporal behavior of the controlled outputs.

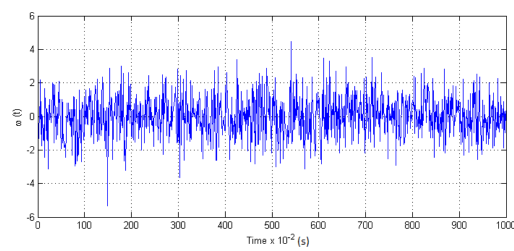

Figure 6: Temporal behavior of the disturbance signal.

5.2. Parametric square SD

Consider the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0{,}1 & 0{,}2 & 1+0{,}1\theta_1 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 0{,}2 \\ -1-\theta_2 & -1-\theta_1 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where $\theta_1 \in [0\ \ 1]$, $\theta_2 \in [0{,}25\ \ 1]$, $\theta_3 \in [0\ \ 0{,}5]$. This system has zero index. Then, for all θ_i, it is regular and impulse free, but is not admissible. Although the matrix $\mathbb{E}$ is uncertain, there are two aspects that allow to apply the design technique that has been developed:

1. For all θ_1, the matrix $\mathbb{E}$ is non-singular (zero index).
2. The matrix of the measured outputs is of complete order, so that the system supports a classic SOF control.

Thus, the transformation linear is $\Gamma = \mathbb{E}^{-1}$, that is

$$\Gamma(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-1}{\theta_1+10} & \frac{-2}{\theta_1+10} & \frac{10}{\theta_1+10} \end{bmatrix}$$

Accordingly, the transformed system corresponds to following LPV system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{5(\theta_1+10)} & 1-\frac{2}{5(\theta_1+10)} & \frac{2}{\theta_1+10} \\ -\theta_2-1 & -\theta_1-1 & 0 \\ \frac{-\theta_3}{\theta_1+10} & \frac{-2\theta_3}{\theta_1+10} & \frac{10\theta_3}{\theta_1+10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1-\frac{1}{\theta_1+10} & -\frac{2}{\theta_1+10} & \frac{10}{\theta_1+10} \\ 1 & 1 & 0 \\ 1-\frac{1}{\theta_1+10} & 1-\frac{2}{\theta_1+10} & \frac{10}{\theta_1+10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The admissibility problem can be solved by stabilizing of transformed LPV system applying the Theorem 11, with $u = \mathcal{K}_0 y$. In effect, the following matrices are obtained:

$$P = \begin{bmatrix} 47{,}2127 & 6{,}8250 & 13{,}3801 \\ 6{,}8250 & 12{,}6129 & -2{,}7575 \\ 13{,}3801 & -2{,}7575 & 10{,}8275 \end{bmatrix},$$

$$X = \begin{bmatrix} -11{,}8221 & 0{,}3080 & -6{,}3564 \end{bmatrix},$$

$$Z = 8{,}9627,$$

then

$$\mathcal{K}_0 = [-1{,}3190 \quad 0{,}0344 \quad -0{,}7092].$$

In order to verify the robust admissibility, Figure 7 shows the location of the poles in closed loop based on the variations of the parameters θ_i, for $i = 1, 2, 3$.

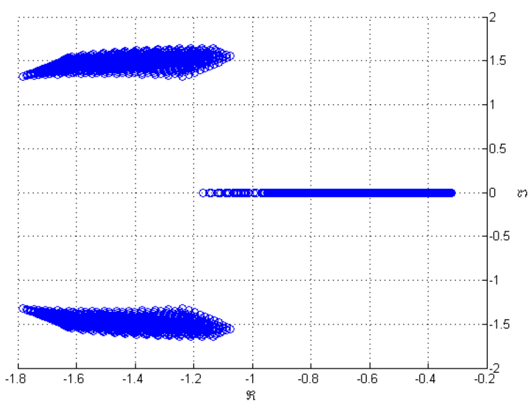

Figure 7: Distribution of the poles for the system in closed loop.

6. Concluding remarks

From the results obtained in this research, the contributions are focused on the analysis and synthesis of controllers for a class of linear descriptor systems dependent on parameters. First, a model of linear descriptor systems with variable parameters has been considered, which consider uncertainties in the descriptor matrix. Then, an analysis of admissibility and robust control for a class of descriptor systems with polytopic parametric uncertainties has been presented. The class is defined by those processes where there is a linear injective application that allows to transform the parameter-dependent descriptor system to a regular LPV system. Thus, the properties and conditions of the original system are conserved in the transformed system, which guarantees the design of a control for the robust admissibility (stability) and the robust performance. The design of the control in the transformed system is a guarantee of satisfying the robust admissibility and performance for the original descriptor system. The synthesis of the control law has been done by means of the static feedback of the extended output, which is based on obtaining a feedback gain for the output and a feedback gain for its derivative. A design technique based on dynamic output feedback has been also proposed, where the dynamic controller is obtained from a static output feedback control problem. The gains are derived by robust stabilizing and robust performance of LPV systems, using the characterizations of the $\mathscr{H}_2$-$\mathscr{H}_\infty$ norms as LMIs, which arise from parameter dependent Lyapunov functions. The design technique also allows to impose multi-objective specifications. The theoretical results have been evaluated through simulations.

Acknowledgements

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