Observations about of the coordinates definition in a vector space and the theorem of Löwig on equicardinality of infinity Hamlet bases

Juan Ojeda, Nelson Hernández*

Mathematics Department, Faculty of Sciences and Technology (FACYT), Carabobo University, Valencia – Carabobo, Venezuela

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Abstract.-

A well known fact perceived in the definition of ordered bases and co- ordinates made in classic texts of Linear Algebra is that there is certain ambiguity or imprecision in the formalization of such concepts, as well as the omission of these definitions for infinite dimensional vector spaces. In this article we reformulate the definition of ordered bases and coordinates for vector spaces, in such a way that this definition adjusts perfectly for finite dimensional vector spaces, as well as for infinite dimensional vector spaces. In addition, a precise revision of the theorem of Löwig on the invariance of dimensionality is presented.

Keywords: infinite dimensional vector spaces, infinite ordered bases, Löwig theorem, dimensionality invariance.

Observaciones acerca de la definición de coordenadas en un espacio vectorial y el teorema de Löwig sobre la equicardinalidad de bases de Hamel infinitas

Resumen.-

Un hecho bien conocido percibido en la definición de bases ordenadas y coordenadas hecho en textos clásicos de Álgebra Lineal es que hay cierta ambigüedad o imprecisión en la formalización de estos conceptos, así como la omisión de estas definiciones para espacios vectoriales de dimensión infinita. En este artículo reformulamos la definición de bases ordenadas y coordenadas de espacios vectoriales, de tal manera que esta definición se ajusta perfectamente a los espacios vectoriales de dimensión finita, así como para los espacios vectoriales de dimensión infinita. Además, se presenta una revisión precisa del teorema de Löwig acerca de la invariancia de la dimensionalidad

Palabras claves: espacios vectoriales de dimensión infinita, bases ordenadas infinitas, teorema de Löwig, invariancia de la dimensionalidad.

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1. Introduction

We will suppose familiarity with the basic concepts of Linear Algebra, as well as some notion of the set theory, especially the Arithmetic of ordinal and cardinal, these concepts can be consulted in [1] and [2]. Then we will follow the standard notation in texts related to these topics; so we will have that:

ω: it will denote the first infinite ordinal.

$(V, F, +, \cdot)$: it will denote a vector space over the field F, with $+$ for the sum of elements of V and $\cdot$ product of a scalar by a vector.

$L(W)$: it will denote the set of all linear combinations of a set W.

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*Autor para correspondencia
Correo-e: `nchernand@uc.edu.ve` (Nelson Hernández)

The ordinals are expressed with the first letters of Greek alphabet and cardinal numbers with the last letters the same alphabet.

We end this introduction highlighting the main purpose of this article is twofold, first to present an updated and more comprehensive version of the theorem Löwig and secondly, to reformulate the concept of ordered basis on a vector space; so to make such formally manageable concept for vector spaces of arbitrary dimension.

2. Preliminary concepts

This section we will remember some fundamental concepts for the development of the present work.

We remember that in some classic texts of Linear Algebra the definition ordered bases is presented (in the finite case) of the following way:

Definition 2.1. *Let* $(V, F, +, \cdot)$ *a finite dimensional vector space, a* `ordered basis` *of* V *is a sequence finite of linearly independent vectors that span* V*.*

In a certain sense, it is not perfectly clear how to formalize this definition, because if $\mathfrak{B} = \{v_1, v_2, \ldots, v_n\}$ is a base of the space, any set whose elements are $v_1, v_2, \ldots, v_n$ will be equal to $\mathfrak{B}$ (by the axiom of extensionalidad of Zermelo-Fraenkel). On the other hand, if the vector space is of infinite dimension not specify the generalization of the concept of ordered bases and coordinates of natural way.

The following definition is related to the generation of subspaces

Definition 2.2. *Let* $(V, F, +, \cdot)$ *a vector space and* $W \subseteq V$*.*

1. *It is said that* $v \in V$ *is a* `linear combination` *of elements of* W *if exists* $S \subseteq W$*, such that* S *is finite and* v *is a linear combination of elements of* S*.*
2. *We say that* V *is* `finitely generated` W*, if* W *is a finite set and* $L(W) = V$*.*
3. *We say that* V *is* `infinitely generated` W*, if* W *is an infinite set and* $L(W) = V$*.*
4. *It is said that* V *is* `strictly infinitely generated` *by* W*, if* V *is infinitely generated by some* W *contained in* V*, and does not exist* S *finite content in* V *it generates.*
5. W *is* `linearly independent` *if all finite subset of* W *is linearly independent.*

Definition 2.3. *Let* $(V, F, +, \cdot)$ *a vector space and* $\mathfrak{B} \subseteq V$*, then we say that* $\mathfrak{B}$ *is an* `infinite Hamel basis` *of* V *if it holds that:*

1. $|\mathfrak{B}| \geq \omega$.
2. $\mathfrak{B}$ *is linearly independent.*
3. V *is strictly infinitely generated by* $\mathfrak{B}$*.*

In this case we will say that $(V, F, +, \cdot)$ *is an infinite dimensional vector space.*

If the space V is finitely generated, a Hamel basis for V is any subset W of V linearly independent, such that $L(W) = V$.

A definition that we will be very useful in the next section is the following:

Definition 2.4. *Let* A *is an infinite set and* κ *a cardinal number, such that* $\kappa \leq |A|$*, then define us and denote us the set of* `all subsets of` A `of cardinality` κ *as*

$$[A]^{\kappa} = \{X \subseteq A : \ |X| = \kappa\}$$

and also the set of `all subsets of` A `of cardinality less than` κ *as*

$$[A]^{<\kappa} = \{X \subseteq A : \ |X| < \kappa\}$$

3. Theorem of Löwig on invariance of dimensionality

It can be proved using Zorn-Kuratowski lemma that all strictly infinitely generated vector space has least one infinite Hamel basis. The demonstration of that two infinite Hamel bases any have the same cardinalidad is not an obvious generalization of the finite case. The following demonstration follows the scheme proposed by Heinrich Löwig, to see [3], detailing and explaining some points which we believe is required.

Theorem 3.1 (Löwig). *Let* $(V, F, +, \cdot)$ *a infinite dimensional vector space and* $\mathfrak{B}$, $\mathfrak{B}'$ *infinite Hamel basis of* V*. Suppose us* $\mathfrak{B} = \{v_\alpha : \alpha < \kappa\}$ *and* $\mathfrak{B}' = \{w_\alpha : \alpha < \lambda\}$ *(where* κ, λ *are infinite cardinal), then* $\kappa = \lambda$*.*

Demonstration:

Define us the following relation R in $\mathfrak{B}$: $v_\alpha R v_\beta$ (with $\alpha, \beta < \kappa$) if and only if exists $I \subset \lambda$ (in where I is a finite set) and a family of nonzero scalars $\{c_i : i \in I\}$, $\{b_i : i \in I\}$, such that

$$v_\alpha = \sum_{i \in I} c_i w_i \qquad \wedge \qquad v_\beta = \sum_{i \in I} b_i w_i$$

Clearly R is an equivalence relation on $\mathfrak{B}$. Let

$$\mathfrak{B}/R = \{[v_\alpha] : \alpha < \kappa\}$$

the set of equivalence classes. Note us that

$$|\mathfrak{B}/R| \leq \kappa$$

We define the function $F : \mathfrak{B}/R \longrightarrow [\lambda]^{<\omega}$, of the following way:

$$F([v_\alpha]) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$$

if and only if exist nonzero scalars (in F) $c_1, c_2, \ldots, c_s$, such that

$$v_\alpha = \sum_{i=1}^{s} c_i w_{\lambda_i}$$

Obviously F is well defined, because it does not depend on the representative of the equivalence class selected. In addition each $\alpha < \lambda$ is in some $B \in Rgo(F)$, but it did not exist $B \in Rgo(F)$ such that $\mu \in B$ we would have that as $\mathfrak{B}$ is a infinite Hamel basis, exists $I \subset \kappa$ (I finite) and nonzero scalars in F $\{c_i : i \in I\}$, in where

$$w_\mu = \sum_{i \in I} c_i v_i.$$

Simultaneously each v_i is expresable in terms of a finite subset of $w's$ (which does not appear w_μ); this contradicts the linear independence of $\{w_\alpha : \alpha < \lambda\}$.

Suppose us

$$|Rgo(F)| = \mu < \lambda$$

then

$$\begin{aligned} \lambda &= |\{w_\alpha : \alpha < \lambda\}| \\ &= \left|\bigcup\{B : B \in Rgo(F)\}\right| \leq \omega \cdot \mu \\ &= \max\{\omega, \mu\} = \mu. \end{aligned}$$

So $|Rgo(F)| = \lambda$ and

$$\lambda = |\mathfrak{B}/R| \leq |\mathfrak{B}| = \kappa$$

Changing roles and defining the equivalence relation of analogous way in $\mathfrak{B}'$, we obtain that $\kappa \leq \lambda$. So applying to the theorem of Cantor-Schröder-Bernstein,, we obtain that $\kappa = \lambda$, that is to say, two infinite Hamel bases of a vectorial space are equipotent.

□

In the light of the previous theorem we can define:

Definition 3.2. *Let* $(V, F, +, \cdot)$ *a vector space. We will say that* V *is of* `Hamel dimension` κ *if there is an infinite Hamel basis* $\mathfrak{B}$*, such that* $|\mathfrak{B}| = \kappa$*, with* κ *infinite.*

This definition also applies to the case where the dimension of the vector space is finite (changing infinite by finite) .

4. Reformulation of the definition of ordered bases and coordinates

In section 1 was realized a brief explanation of the reason for which the definition of ordered bases and coordinates required a revision. We introduce next a reformulation of the definition of these concepts so important in linear algebra:

Definition 4.1. *Let* $(V, F, +, \cdot)$ *a vector space of dimension* κ *(being* κ *finite or infinite) one ordered basis consists of a Hamel base* $\mathfrak{B}$ *of* V*, together with a bijective function* $f : \kappa \longrightarrow \mathfrak{B}$*. The ordered pair* $\widetilde{\mathfrak{B}} = (\mathfrak{B}, f)$ *is called* `ordered basis` V*.*

Considering the fact that each element of a vectorial space can be expressed of form unique as linear combination (finite) of the elements of the base ordinate $\widetilde{\mathfrak{B}}$, then already we are in property to present of coordinates of $v \in V$ with respect to the base ordinate $\widetilde{\mathfrak{B}}$, which we will do of the following form:

Definition 4.2. *Let* $(V, F, +, \cdot)$ *a vector space and* $\widetilde{\mathfrak{B}} = (\mathfrak{B}, f)$ *one ordered base of* V. *Suppose us* $f(\alpha) = v_\alpha$ *(*$\alpha < \kappa$, $v_\alpha \in \mathfrak{B}$*). Then if* $v \in V$, *there is* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \kappa$ *and* $\{c_1, c_2, \ldots, c_n\} \subseteq F$ *(*c_i *nonzero), such that*

$$v = \sum_{i=1}^{n} c_i v_{\alpha_i}$$

Then the `coordinates` v `with respect to the ordered basis` $\widetilde{\mathfrak{B}}$ *is defined as the function* $[v]_{\widetilde{\mathfrak{B}}} : \kappa \longrightarrow F$, *given by*

$$[v]_{\widetilde{\mathfrak{B}}}(\alpha) = \begin{cases} c_i & \text{if } f(\alpha) = v_{\alpha_i}; \\ & (\text{for some } i,\ 1 \le i \le n) \\ & \\ 0 & \text{in other case.} \end{cases}$$

5. Conclusion

This article tries, along with other intentions, to stimulate the study of Linear Algebra in a unified way; introducing the concept of dimension, ordered basis and coordinates of an infinite dimensional vector space; emphasizing the common aspects with the case of a finite dimensional vector space and detailing the differences. Our vision is that such an approach could be useful for the better understanding of the nature of a great variety of vectorial spaces that appear daily in mathematical tasks. It is important to note that the study of infinite dimensional vector spaces is of great relevance in large branches of applied mathematics. To cite just two fundamental examples: Hilbert spaces and Fourier series (which are widely used in physics). So the study of these topics is not exclusively subordinate to pure mathematics.

References

[1] Kenneth Hoffman and Ray Kunze. *Linear Algebra*. Prentice Hall, New Jersey, second edition, 1971.

[2] Thomas J. Jech. *Set Theory*. Academic Press, New York, 1978.

[3] Nathan Jacobson. *Lectures in abstract algebra*, volume I, II, III. Springer–Verlag, New York, 1910.